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OVERSHOOTING AND THE μ-BARRIER

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ABSTRACT

Using general arguments and without numerical calculations, we present three complementary "proofs" that the extent of overshooting (OV) outside a convective region is decreased by a gradient of the mean molecular weight.

Subject headings: convection — stars: evolution — stars: interiors — Sun: evolution — Sun: interior — turbulence

As discussed in an earlier work (Canuto 1997, where relevant references to recent work can be found), we still do not have a theoretical prediction that matches the observational suggestions that both for the Sun (below the convective zone) and for massive stars the "extent of overshooting" (OV) is a small fraction of the local pressure scale height. In massive stars, there is a further consideration: the mean molecular weight μ is an effective barrier to the fluid elements that overshoot into the radiative region. Thus, two barriers actually exist: stable stratification and a μ-gradient (e.g., Ledoux 1947; Schwarzschild & Harm 1958; Kato 1966; Stothers 1970; Langer, Sugimoto, & Fricke 1983; Umezu 1998). Since overshooting is a dynamical consequence of Newton's law and as such is unavoidable, we formulate the problem as follows: given an OV (which cannot be zero) computed as if the medium were homogeneous ($\nabla_\mu = 0, \nabla_\mu \equiv \partial \ln \mu / \ln P$), what is the effect of a μ-gradient? In this Letter we prove the following result:

$$\mathrm{OV}(\nabla_\mu) < \mathrm{OV}(\nabla_\mu = 0). \tag{1}$$

Of the dynamic equations necessary to describe nonlocal convection (Canuto 1997; Canuto & Dubovikov 1998), we employ only the one for the turbulent kinetic energy K:

$$\frac{\partial K}{\partial t} + D_f(K) = g\alpha J - \epsilon. \tag{2}$$

Here, g is the local gravity, α is the thermal expansion coefficient ($\equiv -\rho^{-1}\partial\rho/\partial T$), J is the convective flux F_C in units of $c_P\rho$, $F_C = c_P\rho J = c_P\langle\rho w\theta\rangle$ (c_P and ρ are the specific heat and density), θ and w are the fluctuating components of the temperature and velocity (in the z-direction), and ϵ is the rate of dissipation of K. The key ingredient for the description of overshooting is nonlocality, which in equation (2) is represented by $D_f(K)$, the divergence of the flux of turbulent kinetic energy $F_{\kappa\epsilon}$,

$$D_f(K) = \frac{\partial}{\partial z} F_{\kappa\epsilon}, \quad F_{\kappa\epsilon} = \langle wK \rangle . \tag{3}$$

Equations (2) and (3) are exact since no approximation (closure) is necessary for their derivation (except of course the Boussinesq approximation, which is well satisfied when OV $\ll H_P$). The local, stationary limit of equation (2) yields the MLT (mixing length theory): the right-hand side of equation (2) vanishes, implying that production equals dissipation. Since by Kolmogorov law, $\epsilon = K^{3/2}\Lambda^{-1}$ (Λ is the "size" of the largest eddy), $J \sim K^{3/2}$, which is the MLT expression.

How does equation (*2*) *change in the presence of a μ-gradient?* One can study the problem using the exact equations for the temperature, velocity and arbitrary concentration field C (Canuto 1998). Here, we follow a more physical procedure. Consider the thermal expansion coefficient α that enters equation (2). It originates from the expansion of the density $\rho = \langle\rho\rangle + \rho'$, where

$$\frac{\rho'}{\rho} = -\alpha\theta. \tag{4}$$

In the presence of an additional scalar field like a concentration, one has instead

$$\frac{\rho'}{\rho} = -\alpha\theta + \alpha_C c'', \quad \alpha_C = \left(\frac{\partial \ln \rho}{\partial C}\right)_{P,T}, \tag{5}$$

where C is the mean concentration and c'' is the fluctuating part of the C field. Much as θ combines with the velocity w to give rise to J in equation (2), the new term c'' does likewise and equation (2) becomes

$$\frac{\partial K}{\partial t} + D_f(K) = g\alpha J - g\alpha_C \Psi - \epsilon, \tag{6}$$

where the "concentration flux" Ψ is defined as (H_P is the pressure scale height)

$$\Psi = \overline{c''w} = -K_C \frac{\partial C}{\partial z} = \alpha_C^{-1} H_P^{-1} K_C \nabla_\mu. \tag{7}$$

Here, K_C is the "turbulent diffusion coefficient," which only a full model of turbulence can provide (Canuto 1998) but whose specific form is not needed for our argument. Substituting equation (7) into equation (6), we obtain the desired equation for K:

$$\frac{\partial K}{\partial t} + D_f(K) = g\alpha J - (\epsilon + gK_C H_P^{-1}\nabla_\mu). \tag{8}$$

The physical interpretation is quite clear: ∇_μ increases the dis-

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sipation rate ϵ, thus it increases the sink and lowers K. With less available kinetic energy, one has a *smaller overshooting.* This result alone, which is model independent, proves our basic relation (eq. [1]). However, we can present the result in a more transparent form. Let us combine equation (6) with flux conservation law, $F_r + F_C + F_{\kappa\epsilon} = F_N$ (the source of the total flux is denoted by N, nuclear), where F_r is the radiative flux whose form need not be specified here. Eliminating $J = F_C(c_P\rho)^{-1}$ between equation (8) and the latter relation, the stationary limit of equation (8) becomes a first-order differential equation in $F_{\kappa\epsilon}$ that can be integrated. Calling $r_{1,2}$ the beginning and end points of the entire convective zone (including overshooting) and r_* (with $r_1 < r_* < r_2$) the point where the convective flux F_C goes from positive to negative, signaling the beginning of the OV proper, and introducing the luminosities $L = 4\pi r^2 F$, one derives

$$\frac{g}{c_P}\int_{r_1}^{r_*} T^{-2}|L_N - L_r|dr = \frac{g}{c_P}\int_{r_*}^{r_2} T^{-2}|L_N - L_r|dr + 4\pi\int_{r_1}^{r_2} T^{-1}r^2\rho\epsilon_{\rm eff}dr. \quad (9)$$

Here, the "*effective dissipation*" $\epsilon_{\rm eff}$ is given by

$$\epsilon \to \epsilon_{\rm eff} = \epsilon(1 + gK_C H_P^{-1}\epsilon^{-1}\nabla_\mu) \equiv Q\epsilon > \epsilon. \quad (10)$$

With $\epsilon = 0$, equation (9) becomes the relation first introduced by Roxburgh (1978). However, ϵ cannot be zero. General physical arguments and recent work by Rosvick & VandenBerg (1998) show the importance of ϵ in stellar structure calculations. Since the second integral in the right-hand side of equation (9) is increased by the presence of ∇_μ in $\epsilon_{\rm eff}$, the OV extent $r_2 - r_*$, specifically, the integral from r_* to r_2 need not be large in order to compensate the left-hand side. *Thus,* ∇_μ *decreases the extent of the OV*. This is a second way of proving the physical content of equation (1). Finally, and even more explicitly, we employ a relation for the decay of the velocity w in the OV, due to Unno, Kondo, & Xiong (1985):

$$w(r) = w(r_*)e^{x\ln(P/P_*)}, \quad P < P_*, \quad (11)$$

where r is an arbitrary point in the OV region, $r_* < r < r_2$. The key parameter is x, which can only be provided by a turbulence model. If we consider a polytrope of index m, $P \sim r^{-m}$, then

$$w(r) = w(r_*)\left(\frac{r_*}{r}\right)^{xm}. \quad (12)$$

Since by construction OV $= r_2 - r_* \ll r_2$, if we take $r = r_2$, we have

$$w(r_2) = w(r_*)\left(1 - \frac{\rm OV}{r_2}\right)^{xm} \approx w(r_*)\left(1 - xm\frac{\rm OV}{r_2}\right). \quad (13)$$

Since $w(r_2) = 0$, the extent of the OV $= r_2/xm$ and since $H_P = r_2 m^{-1}$, we finally derive

$$\frac{\rm OV}{H_P} = \frac{1}{x}. \quad (14)$$

From the work of Unno et al., one can deduce that under the scaling $\epsilon \to Q\epsilon$, the variable x scales like

$$x \to Qx, \quad (15)$$

and thus in the presence of a ∇_μ, the decay law (14) changes to

$$\frac{\rm OV}{H_P} = \frac{1}{x}\frac{1}{Q}, \quad (16)$$

and thus, finally,

$$\frac{{\rm OV}(\nabla_\mu)}{{\rm OV}(\nabla_\mu = 0)} = \frac{1}{Q} < 1, \quad (17)$$

which is the desired result: *the* μ*-barrier increases the rate of dissipation of available turbulent kinetic energy and leads to a decrease of the extent of the OV*. Of course, without the solution of the pertinent equations one cannot evaluate Q but that was not the purpose of this Letter. To compute both ϵ and K_C one needs to solve the complete dynamical problem. An attempt in that direction has recently been formulated (Canuto 1998).

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